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MAXIMAL INEQUALITIES RELATED TO GENERALIZED A.E. CONTINUITY

BY

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ABSTRACT. An integral inequality of the classical Hardy-Littlewood type is obtained for the maximal function of positive convolution operators associated with approximations of the identity in R^n. It is shown that the (formally) rearranged maximal function can in general be estimated by an elementary integral involving the decreasing rearrangements of the kernel of the approximation and the function being approximated. (The estimate always holds when the kernel has compact support or a decreasing radial majorant integrable in a neighborhood of infinity; a one-dimensional counterexample shows that integrability alone may not suffice.)

The finiteness of the integral determines a Lorentz space of functions which are a.e. continuous in the generalized sense of the approximation. Conversely, in dimension one it is established that this space is the largest strongly rearrangement invariant Banach space of such functions. In particular, the new inequality provides access to the study of Cesàro continuity of order less than one.

1. Introduction. In this paper we are interested in sharpening Lebesgue's result that when $f \in L^1_{\mathrm{loc}}(R)$, for almost every x:

$$h_0^{-1}\int_0^h |f(x \pm t) - f(x)|\, dt \to 0 \qquad (h \to +0). \tag{1}$$

We interpret (1) as (two-sided) C_1-continuity of f at x determined by the C_1-convergence of $|f(x \pm t) - f(x)|$ to zero as $t \to +0$, and consider weaker summability methods such as C_α $(0 < \alpha \leqslant 1)$, or, more generally, those of Wiener type defined by

$$h^{-1}\int_0^h \varphi(t/h)|f(x \pm t) - f(x)|\, dt \to 0 \qquad (h \to +0), \tag{2}$$

where $0 \leqslant \varphi \in L(0, 1)$ and $\int_0^1\varphi(t)\, dt = 1$.

If φ is increasing, the corresponding method is weaker than C_1 so that (2) is stronger than (1). We use (2) to define (two-sided) φ-continuity of f at x and seek those function spaces in which (2) holds a.e. for each element of the space. It suffices to consider only those f which vanish outside (0, 1). We have particular interest in those spaces which are rearrangement invariant; i.e., in determining sufficient conditions which can be expressed in terms of f^*, the

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decreasing rearrangement of $|f|$. (For our purposes, F^* is defined for any nonnegative F as an inverse of the distribution function determined by outer measure.) We show that the best possible such condition is

$$\int_0^1 \varphi^*(t)f^*(t)\,dt < +\infty, \tag{3}$$

which determines the Lorentz space $\Lambda_\varphi(0, 1)$ [10]. In such spaces, it suffices to establish, say, φ-continuity from the left a.e.

The (left) continuity property (2) for a nonnegative measurable f is determined by convolution operators which approximate the identity; viz.,

$$(T_h f)(x) = h^{-1}\int_0^h \varphi(t/h)f(x-t)\,dt \qquad (h > 0). \tag{4}$$

Through the work of Kolmogorov and Stein [9], [13], it is known that convergence a.e. in a function space should, in general, depend upon weak-type inequalities for the associated maximal function

$$M_\varphi f(x) = \sup_{h>0} T_h f(x). \tag{5}$$

When $\varphi = 1$, this is the famous maximal function Mf of Hardy-Littlewood [13, §10:18] which satisfies the fundamental inequality

$$(Mf)^*(\xi) \leqslant (Mf^*)(\xi) = \int_0^1 f^*(t\xi)\,dt \qquad (0 < \xi \leqslant 1). \tag{6}$$

This inequality implies all of the relevant weak or strong estimates of Mf needed in applications, and it is obviously desirable to find an analogue for $M_\varphi f$. That this is indeed possible constitutes our main result (Theorem 1). In the most interesting case where φ is increasing, it takes the form

$$(M_\varphi f)^*(\xi) \leqslant A(M_\varphi f^*)(\xi) = A\int_0^1 \varphi^*(t)f^*(t\xi)\,dt \tag{7}$$

for $\xi \in (0, 1)$ and an absolute constant $A > 0$ (see Remark 3). The inequality without the middle term is true in general; however, when φ is decreasing it is well known and easy to show that $M_\varphi f$ can be estimated by Mf. The deleted inequality also holds for nonnegative f and φ without compact support, but in the latter case restrictions on φ are essential (Theorem 3, Proposition 4).

We establish the deleted maximal inequality in a multidimensional version which takes the same form and is optimal in various senses but, perhaps, not final. However, our estimate for the multidimensional Hardy-Littlewood maximal function is final within a constant factor and thus constitutes an improvement over the usual weak-type inequality [16] (see also [15]).

For our applications we employ a "positive" version of the Kolmogorov-Stein principle given somewhat abstractly (Theorem 5) in a form which suppresses requirements of linearity and continuity. (It deals with positive operators depending upon a continuous parameter which commute with translations while acting on a translation invariant function space of non-negative functions having a suitable completeness property–viz., the existence of an upper bound.) The resulting weak-type estimate of a restricted maximal function is not only necessary for a.e. convergence, but is essentially sufficient as well, and it is an easy consequence of our maximal inequality (Theorems 4, 6, Corollary 7).

For some recent alternative generalizations of the Hardy-Littlewood maximal inequality with associated applications, see the papers of Burkholder, Caffarel and Calderón, Cordoba, and Fefferman and Stein listed in the references.

2. A general maximal inequality. Let $\mathfrak{M}$ ($\mathfrak{M}^+$) denote the class of (non-negative) measurable extended real valued functions on the Euclidean space of fixed dimension d, and $|E|$ ($|E|_o$), the Lebesgue (outer) measure of a set $E \subseteq \mathbf{R}^d$. For any extended real valued function F defined on $\mathbf{R}^d$, introduce for each $\tau > 0$ the set $E_\tau = \{x \in R^d: |F(x)| > \tau\}$, the resulting distribution function $|E_\tau|_o$, and the decreasing "rearrangement" F^* defined by $F^*(\xi) = \inf\{\tau > 0: |E_\tau|_o \leqslant \xi\}$ $(\xi > 0)$; F^* is seen to be right continuous in the extended sense. See [**11**, §5.4] and [**4**] for further discussion of this definition.

Denote by I the closed cube of unit side in $\mathbf{R}^d$ which is centered at the origin with edges parallel to the coordinate axes.

THEOREM 1. *If $\varphi \in \mathfrak{M}^+$ with support in I, then for each $f \in \mathfrak{M}^+$, the function*

$$M_\varphi f(x) \stackrel{\text{def}}{=} \sup_{h>0} \int \varphi(y) f(x - hy)\, dy \qquad (x \in \mathbf{R}^d)$$

satisfies the following inequality for $\xi > 0$:

$$(M_\varphi f)^*(\xi) \leqslant A \int_0^\infty \varphi^*(At) f^*(t\xi)\, dt \leqslant A \int_0^\infty \varphi^*(t) f^*(t\xi)\, dt \tag{8}$$

for an absolute constant $A \in [1, 3^d]$.

REMARK 1. The second inequality is trivial since $A \geqslant 1$. Monotonicity and simple substitutions yield the following estimates for $\xi > 0$:

$$\min(1, \xi^{-1}) \int_0^\infty \varphi^* f^* \leqslant \int_0^\infty \varphi^*(t) f^*(t\xi)\, dt \leqslant \max(1, \xi^{-1}) \int_0^\infty \varphi^* f^*$$

which show that the integral on the right of (8) is either always finite or always infinite. Thus the maximal inequality is nontrivial only when $f \neq 0$ and φ (or φ^*) is integrable since otherwise $\int_0^\infty \varphi^* f^* = +\infty$.

Proof. It suffices to assume that the right side of (8) is finite for all $\xi > 0$. We first restrict attention to $h \leqslant \delta < +\infty$, indicating this by the superscript (δ) when it is relevant, and for a fixed $\delta > 0$, introduce the sets

$$E_\tau = \left\{x\colon M_\varphi^{(\delta)}f(x) > \tau\right\} \qquad (\tau > 0).$$

Now consider a $\tau > 0$ for which $|E_\tau|_o > 0$. Each $x \in E_\tau$ may be considered as the center of a cube of side $h = h(x) \leqslant \delta$ for which

$$h^d < h^d\tau^{-1}\int \varphi(y)f(x - hy)\,dy = \tau^{-1}\int \varphi(h^{-1}(x - y))f(y)\,dy. \quad (9)$$

By a standard covering principle (e.g., [**14**, §1.6]), a disjoint sequence $\{I_n\}$ of these cubes can be selected such that for an absolute constant $A \in [1, 3^d]$,

$$|E_\tau|_o \leqslant A\sum_n |I_n| = A\sum h_n^d$$

where h_n is the side of I_n with center x_n.

Introducing

$$H_N = \sum_{n \leqslant N} h_n^d \quad \text{and} \quad \varphi_N(y) = \sum_{n \leqslant N} \varphi\big(h_n^{-1}(x_n - y)\big),$$

for $N = 1, 2, \ldots$, we have

$$H_N \leqslant \tau^{-1}\int \varphi_N(y)f(y)\,dy \leqslant \tau^{-1}\int_0^\infty \varphi_N^*(t)f^*(t)\,dt,$$

but since the terms of the defining sum for φ_N have disjoint support, for each $\xi > 0$:

$$\begin{aligned}\varphi_N^*(\xi) &= \inf\left\{\alpha > 0\colon \sum_{n \leqslant N} |y\colon \varphi\big(h_n^{-1}(x_n - y)\big) > \alpha| \leqslant \xi\right\}\\ &= \inf\left\{\alpha > 0\colon \sum_{n \leqslant N} h_n^d|\varphi > \alpha| \leqslant \xi\right\}\\ &= \inf\{\alpha > 0\colon |\varphi > \alpha| \leqslant \xi/H_N\} = \varphi^*(\xi/H_N).\end{aligned}$$

Hence

$$\tau \leqslant H_N^{-1}\int_0^\infty \varphi^*(t/H_N)f^*(t)\,dt = \int_0^\infty \varphi^*(t)f^*(tH_N)\,dt.$$

Now, as $H_N \nearrow H \geqslant |E_\tau|_o/A$, there follows from the monotonicity of f^* and dominated convergence the estimate

$$\begin{aligned}\tau &\leqslant \int_0^\infty \varphi^*(t)f^*(tH)\,dt \leqslant \int_0^\infty \varphi^*(t)f^*(t|E_\tau|_o/A)\,dt\\ &\leqslant A\int_0^\infty \varphi^*(At)f^*(t|E_\tau|_o)\,dt,\end{aligned}$$

which requires only the positivity of $|E_\tau|_o$ and admits the possibility that $|E_\tau|_o$ (or H) is infinite.

Next, if for some $\xi > 0$, $\tau_0 = (M_\varphi^{(\delta)}f)^*(\xi)$ $(= \inf\{\tau > 0\colon |E_\tau|_o < \xi\})$ is positive, then when $0 < \tau < \tau_0$, $|E_\tau|_o > \xi > 0$, and so from the preceding inequality

$$\tau \leq A\int_0^\infty \varphi^*(At)f^*(t|E_\tau|_o)\,dt \leq A\int_0^\infty \varphi^*(At)f^*(t\xi)\,dt;$$

hence as $\tau \nearrow \tau_0$, there follows the estimate

$$\tau_0 = \left(M_\varphi^{(\delta)}f\right)^*(\xi) \leq A\int_0^\infty \varphi^*(At)f^*(t\xi)\,dt.$$

Finally as $\delta \to +\infty$, $M_\varphi^{(\delta)}f \to M_\varphi f$ pointwise and the continuity of outer measure gives the corresponding pointwise convergence of the "rearrangements"; the desired inequality (8) follows.

REMARK 2. The constant A appearing in (8) is precisely that required for the covering principle and is thus universal for all φ with support in I. If instead φ has support in cI for some $c > 0$, it is straightforward to verify that the corresponding inequality with the *same* A will be as follows:

$$\begin{aligned}(M_\varphi f)^*(\xi) &\leq Ac^d\int_0^\infty \varphi^*(Ac^dt)f^*(t\xi)\,dt\\ &\leq Ac^d\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt \qquad (\text{if } c \geq 1).\end{aligned}$$

For more specialized φ and f, the covering argument of Walker [15] may be used to obtain $A = 2^d$, in which case our upper estimates for $M_\varphi f$ both agree with and extend those obtained by him. In dimension one, $A = 1$ is achievable for a large class of φ. The analysis is rather involved and will be deferred to a subsequent paper. Extensions to φ without compact support will be considered in the next two sections.

REMARK 3. In dimension one, the maximal inequality is optimal in the following sense:

When $0 \leq \varphi$ increases and $0 \leq f$ decreases on $J = (0, 1)$ and both vanish elsewhere, then for $\xi \in J$,

$$(M_\varphi f)(\xi) = \int_0^1 \varphi^*(t)f^*(t\xi)\,dt = (M_\varphi f)^*(\xi); \tag{10}$$

hence, for such φ, the integral in (8) equals $(M_\varphi f^*)(\xi)$ for *arbitrary* $f \in \mathfrak{M}^+$ with support in J. (See also the discussion of (7) in the introduction.)

To prove (10), observe that using monotonicity and the supports of φ and f we have for $\xi > 0$ the inequalities

$$\int_0^1 \varphi(t)f(\xi - th)\,dt \leq \int_0^1 \varphi(t)f(\xi - t\xi)\,dt \qquad (0 < h \leq \xi)$$

and

$$h^{-1}\int_0^{\xi} \varphi(th^{-1})f(\xi - t)\, dt \leq \xi^{-1}\int_0^{\xi} \varphi(t\xi^{-1})f(\xi - t)\, dt \qquad (h \geq \xi);$$

thus the maximum over h is attained when $h = \xi$. This establishes the left equality in (10) and the right follows from the observation that the integral is a decreasing continuous function of ξ.

3. Extensions of the maximal inequality. If instead of $M_\varphi f$ we consider for $f \in \mathfrak{M}^+$,

$$L_\varphi f(x) \overset{\text{def}}{=} \overline{\lim_{h \searrow 0}} \int \varphi(y) f(x - hy)\, dy$$

then the more precise Vitali covering theorem can be used to replace the covering principle employed in the proof of Theorem 1. As a result, the constant A can be replaced by unity, and except for the corresponding changes, the same proof yields

THEOREM 2. *If $\varphi \in \mathfrak{M}^+$ with support in I, and $f \in \mathfrak{M}^+$ then*

$$(L_\varphi f)^*(\xi) \leq \int_0^{\infty} \varphi^*(t) f^*(t\xi)\, dt \qquad (\xi > 0).$$

We next relax the requirement that φ have compact support. For $\varphi \in \mathfrak{M}^+$ without compact support, let

$$\tilde{\varphi}(x) \overset{\text{def}}{=} \sup_{|y| \geq |x|} \varphi(y) \qquad (x \in \mathbf{R}^d)$$

define its decreasing radial majorant $\tilde{\varphi}$. We say that $\tilde{\varphi}$ is integrable at infinity if for some $T > 0$, $\int_{|x| \geq T} \tilde{\varphi}(x)\, dx < +\infty$. (The importance of $\tilde{\varphi}$ for convergence a.e. was first demonstrated by Calderón and Zygmund [3] who considered the case when $\tilde{\varphi}$ is integrable over $\mathbf{R}^d$.)

THEOREM 3. *If $\varphi \in \mathfrak{M}^+$ has a decreasing radial majorant $\tilde{\varphi}$ which is integrable at infinity, then $\exists A(\varphi) < +\infty$ such that $\forall f \in \mathfrak{M}^+$:*

$$(M_\varphi f)^*(\xi) \leq A(\varphi) \int_0^{\infty} \varphi^*(t) f^*(t\xi)\, dt \qquad (\xi > 0). \tag{11}$$

PROOF. From the hypotheses, we may express $\varphi = \varphi_1 + \varphi_2$ where $\varphi_k \in \mathfrak{M}^+$, $k = 1, 2$,

$$\begin{aligned} \varphi(x) &= \varphi_1(x), \quad |x| \leq l, \\ &\qquad\qquad\qquad\qquad \text{for some } l > 0, \\ &= \varphi_2(x), \quad |x| > l, \end{aligned}$$

and φ_1 has compact support while φ_2 has an integrable bounded decreasing radial majorant; i.e.

$$0 \leq \varphi_2(x) \leq \rho(|x|) \quad \forall x,$$

$0 < \rho$ is decreasing on $[0, \infty)$ and $U = \int_0^\infty t^{d-1}\rho(t)\,dt < +\infty$. We can further suppose (by a scale change if necessary) that $\varphi^*(\omega) > 0$ where ω denotes the measure of the unit ball in $\mathbf{R}^d$. (The next estimate of $M_{\varphi_2}f$ is derived for the sake of completeness; see [**14**, §2.2].) We may further suppose that ρ is left continuous and use it to define the Borel measure μ on $[0, \infty)$ by $\rho(t) = \mu[t, \infty)$ $(t > 0)$. Then for $s > 0$ and a suitable dimensional constant c:

$$\int_s^{2s} t^d d\mu(t) \leqslant (2s)^d \rho(s) \leqslant c \int_{s/2}^s t^{d-1}(t)\,dt,$$

and, taking $s = 2^n$, $n = 0, \pm 1, \pm 2, \ldots,$ we get $\int_0^\infty t^d\,d\mu(t) \leqslant cU$. Hence, by the Fubini theorem, for each $x \in \mathbf{R}^d$ and $h > 0$:

$$h^{-d}\int \varphi_2(h^{-1}(x-y))f(y)\,dy \leqslant \int_0^\infty t^d\,d\mu(t)(th)^{-d}\int_{|x-y|<th} f(y)\,dy,$$

so that $M_{\varphi_2}f(x) \to cUM_\sigma f(x)$, where σ denotes the characteristic function of the unit ball.

Clearly

$$M_\varphi f(x) \leqslant M_{\varphi_1}f(x) + M_{\varphi_2}f(x) \leqslant M_{\varphi_1}f(x) + cUM_\sigma f(x),$$

and a double application of Theorem 1 (Remark 2) with standard arguments gives for each $\xi > 0$ the estimate

$$\begin{aligned}(M_\varphi f)^*(2\xi) &\leqslant 2(M_{\varphi_1}f)^*(\xi) + 2cU(M_\sigma f)^*(\xi)\\ &\leqslant A(\varphi_1)\int_0^\infty \varphi_1^*(t)f^*(t\xi)\,dt + A(\varphi_2)\int_0^\omega f^*(t\xi)\,dt\\ &\leqslant A(\varphi_1)\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt + A(\varphi)\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt\end{aligned}$$

where the obvious inequality $\varphi_1^*(t) \leqslant \varphi^*(t)$ was used in the first integral while the second requires only the assumed positivity of $\varphi^*(\omega)$. The desired inequality follows from the estimate $\varphi^*(t) \leqslant \varphi^*(t/2)$ $(t > 0)$ and obvious substitutions.

REMARK 4. From the discussion in Remark 1 it is clear that the inequality of Theorem 3 is trivial when φ^* is not integrable at zero. That the inequality *cannot* hold when φ^* is locally but not globally integrable on $[0, \infty)$ is shown by taking $f = \sigma$. Then the integral in (11) is $\int_0^{\omega/\xi}\varphi^*(t)\,dt$ which is finite for $\xi > 0$ by assumption.

On the other hand,

$$\int \varphi(y)f(x-hy)\,dy = \int_{|h^{-1}x-y|<h^{-1}} \varphi(y)\,dy$$

and for each x for which $|x| < 1$, the latter integral can be made for sufficiently small h as near $\int \varphi(y)\,dy = +\infty$ as desired. Hence $(M_\varphi f)^*(\xi) = +\infty$ on an open interval and this violates the inequality.

For integrable $\varphi^* \neq 0$, it follows that $A(\varphi) \geqslant 1$. Indeed we may suppose $\int_0^\infty \varphi^* = 1$, and then with $f = \sigma$ as above, clearly $M_\varphi f(x) \leqslant 1$ everywhere while $M_\varphi f(x) \equiv 1$ for $|x| < 1$. Hence for some $\xi > 0$,

$$(M_\varphi f)^*(\xi) = 1 \leqslant A(\varphi)\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt \leqslant A(\varphi).$$

The same discussion is also applicable to $L_\varphi f$ (see Theorem 2).

4. Limitations of the maximal inequality. As we have just shown, integrability of φ^* (or φ) is necessary for the maximal inequality of Theorem 3 to hold and be nontrivial. That integrability alone does not suffice is illustrated by the following counterexample in $\mathbf{R}^1$ in which φ and f are taken to be even step functions on $(-\infty, +\infty)$; the nonzero values of φ are bounded and decrease but not rapidly enough to admit an integrable $\tilde{\varphi}$, while those of f grow just rapidly enough to force $M_\varphi f = +\infty$. Specifically, we have

PROPOSITION 4. *Let $\rho(x)$ be defined, positive and increasing for $x > x_0 > e^3$ with the additional limiting properties that as $x \to +\infty$:*
(i) $\rho(x)/(\log^2 x \log_2^{1+\varepsilon} x) \to \infty$ *for some* $\varepsilon > 0$,
(ii) $\rho(x)/\log^3 x$ *decreases to zero,*
(iii) $\rho(x+1)/\rho(x) \to 1$.
Then there exist integrable even nonnegative step functions φ with $\varphi(x) \leqslant 2x^{-1}\rho(x)$ $(x > x_0)$, and f, for which $L_\varphi f(x) = +\infty$ $\forall x \in R$ while $\int_0^\infty \varphi^* f^* < +\infty$, so that the maximal inequalities of Theorems 2 *and* 3 *fail* $\forall \xi > 0$.

PROOF. For positive integers $n \geqslant n_0 > x_0$ set $\delta(n) = (\rho(n)\log^2 n)^{-1}$ and define

$$\varphi(x) = \varphi(|x|) = n^{-1}\rho(n), \qquad |x| \in [n - \delta(n), n + \delta(n)],$$
$$= 0 \quad \text{otherwise.}$$

Here n_0 is to be chosen large enough to have $0 \leqslant \varphi \leqslant 1$, $\delta(n) < \frac{1}{2}$, $n\delta(n) \geqslant 1$ and $\varphi(n) \leqslant 2(n+1)$ when $n \geqslant n_0$; these conditions assure for φ the properties required for the proposition. Similarly for $n > 0$ set

$$f(x) = f(|x|) = c(n)e^n, \qquad |x| \in [n - e^{-n}, n + e^{-n}],$$
$$= 0 \quad \text{otherwise,}$$

where $c(n) > 0$ is to be chosen to have $\sum_n c(n) < +\infty$ while $nc(n)\log^{1+\varepsilon} n \to +\infty$ (e.g., $c(n) = (n \log^{1+\varepsilon/2} n)^{-1}$). Since $0 \leqslant \varphi \leqslant 1$, $\varphi^* \leqslant 1$; and for $\xi > 0$:

$$\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt \leqslant \xi^{-1}\int_0^\infty f^* = 4\xi^{-1}\sum_n c(n) < +\infty.$$

On the other hand, by Dirichlet (or continued fractions) for each $x \geqslant 0$, there are infinitely many integers $q > n_0$ such that $qx = p + q^{-1}\theta$ for some nonnegative integer p and $|\theta| \leqslant 1$. For each such q, let $n = n(q) = [\log(q/\delta(q))]$ so that $q^{-1} \leqslant \delta(q) \leqslant qe^{-n}$. Then when q is so large that

$(n(q) - x)q > n_0 + 1$, it follows that $nq - p > n_0$ for $p = p(q)$ as above. Hence for $h = q^{-1}$, utilizing the symmetry of φ there follows the lower estimate:

$$M_\varphi f(x) \geq \int_{-\infty}^{\infty} \varphi(t - qx)f(t/q)\,dt$$
$$\geq \delta(nq)\varphi(nq)f(n) = c(n)e^n/\left(nq \log^2(nq)\right),$$

since $\varphi(t - qx) = \varphi(nq - p) \geq \varphi(nq)$ on a subinterval of length $\delta(nq) \leq \delta(q) \leq qe^{-n}$ subordinate to the interval of length $2qe^{-n}$ on which $f(t/q) = f(n)$. (Observe that the relevant intervals supporting φ and f have centers differing by $|\theta/q| \leq q^{-1} \leq qe^{-n}$.)

Since $e^{n+1} > q/\delta(q)$, the right side of the above inequality exceeds

$$\frac{1}{\varepsilon}\,\frac{c(n)\rho(q)\log^2 q}{n\log^2(nq)} \sim \frac{c(n)n\log^{1+\varepsilon}n}{e}\,\frac{\rho(q)}{\log^2 q\,\log_2^{1+\varepsilon} q}$$

where we have utilized the hypotheses on ρ together with the definition of $n(q)$ to get the asymptotic formula on the right which approaches $+\infty$ with q by the assumed conditions on ρ and c.

Since $M_\varphi f$ is even, it follows that $M_\varphi f(x) = +\infty\ \forall x$. However because the h selected $\searrow 0$ as $q \to +\infty$, the same proof shows that $L_\varphi f(x) = +\infty\ \forall x$.

REMARK 5. A possible choice for ρ in the proposition is $\rho(x) = \log^{2+\varepsilon}x$, $x > x_0$ $(0 < \varepsilon < 1)$. However, it seems difficult to achieve $\rho(x) = \log^2 x$, while the choice $\rho(x) = 1$ cannot provide a counterexample through the arguments used in this construction. Indeed, the lower estimate for $M_\varphi f(x)$ requires that with $2\gamma(n) = |t: f(t) = f(n)|$, we must have $\delta(nq) \leq 2q\gamma(n)$; hence

$$\delta(nq)\varphi(nq)f(n) \leq 2n^{-1}f(n)\gamma(n) \leq 2f(n)\gamma(n),$$

which must $\to 0$ as $n \to +\infty$ whenever f is integrable.

5. φ-continuity almost everywhere.

DEFINITION. If $\varphi \in \mathfrak{M}^+$ with compact support and $\int_0^\infty \varphi^* = 1$, we say that $f \in \mathfrak{M}$ is (left) φ-continuous at $x \in \mathbf{R}^d$ when $|f(x)| < +\infty$ and

$$\lim_{h \searrow 0} \int \varphi(y)|f(x - hy) - f(x)|\,dy = 0;$$

it follows that

$$\lim_{h \searrow 0} h^{-d}\int \varphi(y/h)f(x - hy)\,dy = f(x).$$

In this section we will employ the maximal inequality of Theorem 2 to investigate φ-continuity a.e. Recall that the maximal integral $\int_0^\infty \varphi^*(t)f^*(t\xi)\,dt$ is finite for $\xi > 0$ *iff* $\int_0^\infty \varphi^* f^* < +\infty$ (Remark 1). On the other hand Lorentz has shown [10] that

$$\Lambda_\varphi = \Lambda_{\varphi^*} = \left\{ f \in \mathfrak{M} : \lambda_\varphi(f) = \int_0^\infty \varphi^* f^* < +\infty \right\}$$

forms a rearrangement invariant Banach space with rearrangement invariant norm λ_φ. Since $\Phi(\xi) \overset{\text{def}}{=} \int_0^\xi \varphi^* > 0$ when $\xi > 0$ (and $\int_0^\infty \varphi^* = 1$), we have the useful estimate

$$f^*(\xi) \leq (\Phi(\xi))^{-1} \int_0^\xi \varphi^* f^* \leq \lambda_\varphi(f)/\Phi(\xi) \qquad (\xi > 0). \tag{12}$$

It follows that each $f \in \Lambda_\varphi$ is finite a.e. and each set $\{|f| > \tau\}$ has finite measure. Standard approximation techniques show that the continuous functions are dense in Λ_φ.

THEOREM 4. *Each $f \in \Lambda_\varphi$ is a.e. (left) φ-continuous.*

REMARK 6. Since Λ_φ is rearrangement invariant we also obtain (right) φ-continuity a.e., where $f(x + y)$ replaces $f(x - y)$.

PROOF. The assertion is trivial if f is itself continuous a.e.; otherwise from the above discussion we may express $f = \psi + g$ where ψ is continuous with compact support and $\lambda_\varphi(g)$ is controllably small. Then

$$\begin{aligned} \Omega f(x) &\overset{\text{def}}{=} \overline{\lim_{h \searrow 0}} \int \varphi(y) |f(x - hy) - f(x)| \, dy \\ &\leq \Omega\psi(x) + \Omega g(x) = \Omega g(x) \\ &\leq (L_\varphi |g|)(x) + |g|(x). \end{aligned}$$

Hence for $0 < \xi \leq 1$, we have from Theorem 2, Remark 1 and (12) that

$$(\Omega f)^*(2\xi) \leq 2(L_\varphi |g|)^*(\xi) + 2g^*(\xi) \leq 2\xi^{-1}\lambda_\varphi(g) + (2\lambda_\varphi(g)/\Phi(\xi))$$

which $\to 0$ as $\lambda_\varphi(g) \to 0$ for each fixed $\xi > 0$. Thus $(\Omega f)^* = 0$ so that $\Omega f = 0$ a.e. and the theorem is proven.

REMARK 7. For each $p > 0$, the same techniques can be used to show that each function in

$$\Lambda_\varphi^p \overset{\text{def}}{=} \left\{ f \in \mathfrak{M} : \int_0^\infty \varphi^* (f^*)^p < +\infty \right\}$$

is a.e. φ-continuous in the mean of order p: i.e.,

$$|f(x)| < +\infty \text{ and } \lim_{h \searrow 0} \int \varphi(y) |f(x - hy) - f(x)|^p \, dy = 0.$$

6. Maximal inequalities of weak-type for positive operators. It is obvious from the definition given in the preceding section that if f is a.e. φ-continuous, then so is each translate of f; since the pioneering work of Kolmogorov [9], it has been recognized that pointwise a.e. convergence in the presence of suitable invariance with translation provides a means of supplying a family of operators with uniform estimates of weak-type. In this section we shall explore the consequences of a.e. φ-continuity from this point of view.

Since it is sufficient to study the local behavior of the nonnegative functions on $\mathbf{R}^d$, we introduce

$\mathcal{P}^+ = \{f \in \mathfrak{M}^+ : f$ is multiperiodic with unit periodic in each coordinate$\}$,

and let $|E|_I = |E \cap I|$ for each periodic measurable set E.

By examining relevant properties of Λ_φ and λ_φ introduced in the previous section, we are led to consider an abstract pair (Λ^+, λ) with the following properties:

(1) Λ^+ is a subset of $\mathcal{P}^+$ which is closed under translation and multiplication by positive scalars.

(2) λ is a nonnegative real valued function on Λ^+ which is homogeneous and translation invariant.

(3) Λ^+ is order complete with respect to λ in the following sense: If $f_n \in \Lambda^+$, $n = 1, 2, \ldots$, and $\Sigma_n \lambda(f_n) < +\infty$, then $\exists\, F \in \Lambda^+$ such that each $f_n \leqslant F$ a.e., $n = 1, 2, \ldots$; i.e., the f_n have an a.e. upper bound in Λ^+. A simple example is furnished by $\Lambda^+ = \mathcal{P}^+ \cap L^1_{\text{loc}}(R)$ with $\lambda(f) = \int_I f$. Observe that Λ^+ is not required to have an additive structure.

We also abstract the corresponding uncountable family of operators required to describe φ-continuity (see (4)) and assume that to each $h \in (0, 1)$ there is assigned an operator T_h: $\Lambda^+ \to \mathcal{P}^+$ which is homogeneous, commutes with translations, and preserves order in the sense that when $f, \tilde{f} \in \Lambda^+$ and $f \leqslant \tilde{f}$ a.e. then $T_h f \leqslant T_h \tilde{f}$ everywhere. Moreover, it is required that for each $f \in \Lambda^+$, $T_h f(x)$ should be measurable in the *pair* (x, h).

It then follows that for each $\delta \in (0, 1)$, the functions defined pointwise by

$$T_0 f(x) = \operatorname*{ess\,sup}_{h<\delta} T_h f(x)$$

and

$$Tf(x) = \lim_{\delta \searrow 0} T_\delta f(x) = \lim \operatorname*{ess\,sup}_{h \searrow 0} T_h f(x)$$

are in $\mathfrak{M}^+$, when "ess" means that h-sets of measure zero are neglected. [Indeed, for a.e. x: $T_h f(x)$ is measurable in h, and thus so is $\chi_\tau(x, h)$, the characteristic function of the set $\{(x, h)$: $T_h f(x) > \tau\}$, for *every* $\tau \geqslant 0$; but for each $\delta \in (0, 1)$,

$$\{x\colon T_\delta f(x) \leqslant \tau\} \text{ agrees a.e. with } \left\{x\colon \int_0^\delta \chi_\tau(x, h)\, dh = 0\right\}$$

which is measurable by Fubini. Thus $T_\delta f$ is measure as is $Tf = \lim_{n\to\infty} T_{1/n} f$.]

Theorem 5. *Let* (Λ^+, λ) *and* T_h, T_δ, T *be as specified above, and suppose that for each* $f \in \Lambda^+$, Tf *is finite on a set of positive measure; then there exist constants* $\delta \in (0, 1)$ *and* $A > 0$ *such that for all* $f \in \Lambda^+$ *and* $\tau > 0$:

$$|T_\delta f > \tau|_I \leqslant A\tau^{-1}\lambda(f).$$

REMARK 8. The operators T_h need not be linear, or even sublinear, nor is there any explicit requirement of continuity. However, they are positive and the conclusion shows that for $h \leq \delta$, they must be in fact uniformly continuous (at 0) in measure with respect to λ. The proof which follows is a modification of an argument in [12].

PROOF. If the conclusion is false, then by homogeneity for $\delta_n = 1/n$ and $A_n = n^2$, $\exists f_n \in \Lambda^+$ with $E_n = \{T_{\delta_n} f_n > 1\}$ and $|E_n|_I > n^2\lambda(f_n)$ for each $n = 1, 2, 3, \ldots$. By repetitions if necessary we can suppose that $\Sigma_n |E_n|_I = +\infty$ while $\Sigma_n p_n \lambda(f_n) < +\infty$ for positive numbers $p_n \nearrow +\infty$ as $n \to \infty$. From a result of Kolmogorov and Stein [13], it follows that for suitable choice of $y_n \in I$, a.e. x is contained in an infinite number of the "left" translates of the sets E_n by y_n.

Set $g_n(x) = f_n(x + y_n)$ and observe that from homogeneity $\Sigma_n \lambda(p_n g_n) = \Sigma_n p_n \lambda(g_n) < +\infty$, so that by the assumed completeness of Λ^+, $\exists F \in \Lambda^+$ with $p_n g_n \leq F$ a.e., $n = 1, 2, \ldots$.

The hypotheses imply that for some positive integer k we have $T_{\delta_k} F \leq k$ on a set E of positive measure. Hence, for $h \leq \delta_n \leq \delta_k$, and $x \in E$:

$$(T_h p_n f_n)(x + y_n) = (T_h p_n g_n)(x) \leq T_h F(x) \leq k,$$

so that $p_n(T_{\delta_n} f_n)(x + y_n) \leq k$ when $n \geq k$, $x \in E$. But this contradicts the fact that for a.e. x, $(T_{\delta_n} f_n)(x + y_n) > 1$ for infinitely many values of n while $p_n \nearrow +\infty$.

REMARK 9. The inequality of Theorem 5 may be expressed in the following equivalent form:

$$|T_\delta f > 1|_I \leq A\lambda(f), \quad \forall f \in \Lambda^+;$$

in this form the theorem would also hold if λ instead of being homogeneous satisfies on Λ^+ say an inequality the form $\lambda(af) \leq a^p \lambda(f)$ $(a > 0)$ for some fixed $p > 0$. Moreover, the operators T_h could be defined only for h in a Borel set $H \subseteq (0, 1)$ on which there is a Borel measure μ such that $\mu((0, \delta] \cap H) > 0$ when $\delta > 0$, providing that for each $f \in \Lambda^+$, $T_h f(x)$ remains measurable with respect to the pair (x, h). In particular, the theorem would hold for a sequence of operators, the usual domain of theorems of this type (see, e.g., [17, p. 165]; [7], [13]).

7. Optimal conditions for a.e. φ-continuity. It is desirable to find maximal function spaces of a.e. φ-continuous functions. Since a.e. φ-continuity is clearly a local property when φ has compact support, it suffices to restrict the functions to the standard open unit cube $J = (0, 1)^d \subseteq \mathbf{R}^d$. However for intrinsic technical reasons, the sufficiency argument of §5 and the necessity theorem of §6 are best formulated on $\mathfrak{M}^+$ and $\mathfrak{P}^+$ respectively. In this section we will bring them to bear simultaneously on the local question of a.e. φ-continuity.

Let $\mathfrak{M}^+(J)$ denote the nonnegative measurable functions on J which are assumed to be extended to zero elsewhere and set $\Lambda_\varphi^+(J) = \Lambda_\varphi \cap \mathfrak{M}^+(J)$ which by Theorem 4 contains only a.e. φ-continuous functions.

The pair $(\Lambda_\varphi^+(J), \lambda_\varphi)$ has the properties which can be stated as follows for the abstract pair (Λ_J^+, λ_J):

$\Lambda_J^+ \subseteq \mathfrak{M}^+(J)$ is closed under multiplication by positive scalars and equimeasurable rearrangement within J, and is complete with respect to the homogeneous rearrangement invariant functional λ_J: $\Lambda_J^+ \to [0, \infty)$ in the sense of §6. [When $(\Lambda_J^+, \lambda_J) = (\Lambda_\varphi^+(J), \lambda_\varphi)$, this last condition is verified as follows:

If $f_n \in \Lambda_\varphi^+(J)$, $n = 1, 2, \dots,$ and $\Sigma_n \lambda_\varphi(f_n) < +\infty$ then $F = \Sigma_n f_n$ is clearly an upper bound for the f_n and it is in $\Lambda_\varphi^+(J)$ since

$$\int_0^\infty \varphi^* F^* = \lim_{N \to +\infty} \int_0^\infty \varphi^* \left(\sum_1^N f_n \right)^* < \sum_n \lambda_\varphi(f_n)$$

by monotone convergence and standard arguments.]

Now, we would like to use Theorem 5 to assert that the largest such Λ_J^+ of a.e. φ-continuous functions is $\Lambda_\varphi^+(J)$. This is at least the case in dimension one for functions φ which are increasing on $J = (0, 1)$. In particular, it is the case for $\varphi(t) = \alpha(1 - t)^{\alpha - 1}$ on J $(0 < \alpha < 1)$ which corresponds to the C_α continuity discussed in the introduction.

THEOREM 6. *Let $\varphi \in \mathfrak{M}^+(J)$ be increasing on $J = (0, 1)$ with $\int_0^1 \varphi = 1$, and (Λ_J^+, λ_J) be a pair as described above. Then if the functions in Λ_J^+ are a.e. φ-continuous,*

$$\Lambda_J^+ \subseteq \Lambda_\varphi^+(J) \quad \textit{and} \quad \exists \textit{ a constant } A > 0$$

such that

$$\lambda_\varphi(g) < A\lambda_J(g) \quad \forall g \in \Lambda_J^+.$$

PROOF. Let Λ^+ denote the class of periodic extensions (f) of the restrictions to $\bar{J}$ of the functions $f_J \in \Lambda_J^+$. Then since Λ_J^+ is closed under rearrangements within J, it follows that Λ^+ is closed under translations. Moreover defining $\lambda(f) = \lambda_J(f_J)$, $f \in \Lambda^+$, it is seen that the pair (Λ^+, λ) has the properties required for Theorem 5, as do the operators T_h: $\mathcal{P}^+ \to \mathcal{P}^+$ defined for $h \in J$ by

$$T_h f(x) = \int_0^1 \varphi(y) f(x - hy)\, dy = h^{-1} \int_0^h \varphi(y/h) f(x - y)\, dy.$$

The joint measurability is a consequence of Fubini, and, in addition, the a.e. continuity of φ makes $T_h f$ lower semicontinuous in h (Fatou). Thus for each $\delta \in J, f \in \Lambda^+$:

$$T_\delta f = \operatorname*{ess\,sup}_{h < \delta} T_h f = \sup_{h < \delta} T_h f.$$

Finally, the hypothesized a.e. φ-continuity of the functions in Λ_J^+ assures that of their periodic extensions in Λ^+, and hence, the finiteness a.e. of $Tf = \overline{\lim}_{h \searrow 0} T_h f$ for each $f \in \Lambda^+$.

Theorem 5 is applicable and provides a $\delta \in J$ for which $\sup_{h<\delta} T_h f$ is finite a.e. for $f \in \Lambda^+$, and thus for a.e. $x \in (0, \delta)$:

$$\int_0^1 \varphi(1-t) f_J(t)\, dt = \int_0^1 \varphi(1-t) f(tx)\, dt$$
$$= T_x f(x) \leq T_\delta f(x) < +\infty,$$

or since φ is increasing on J while $f_J^* \in \Lambda_J^+$, for a.e. $x \in (0, \delta)$:

$$\int_0^1 \varphi^*(t) f_J^*(t)\, dt \leq \int_0^1 \varphi^*(t) f_J^*(tx)\, dt \leq T_\delta f_{\text{per}}^*(x) < +\infty$$

where f_{per}^* devotes the periodic extension of f_J^*.

Thus $\Lambda_J^+ \subseteq \Lambda_\varphi^+(J)$.

Next, utilizing the full conclusion of Theorem 5, $\exists A > 0$ such that if $0 < \tau < \lambda_\varphi(f_J)$ for some $f_J \in \Lambda_J^+$, we have from the above inequalities

$$\delta = \left| x \in (0, \delta): \int_0^1 \varphi^*(t) f_J^*(tx)\, dt > \tau \right|$$
$$\leq |T_\delta f_{\text{per}}^* > \tau|_I \leq A\tau^{-1}\lambda(f_{\text{per}}^*).$$

Thus

$$\tau \leq A\delta^{-1}\lambda(f_{\text{per}}^*) = A\delta^{-1}\lambda_J(f_J^*) = A\delta^{-1}\lambda_J(f_J)$$

and the proof is completed by allowing $\tau \nearrow \lambda_\varphi(f_J)$.

For φ as in Theorem 6, the Banach space

$$\Lambda_\varphi(J) = \{ f \in \mathfrak{M}: |f| \in \Lambda_\varphi^+(J) \}$$

is "strongly rearrangement invariant" in the sense that a measurable $f \in \Lambda_\varphi(J)$ *iff* $f^* \in \Lambda_\varphi(J)$. This concept extends to Banach function spaces, B, and their norms: for measurable f, $f \in B$ iff $f^* \in B$, $\|f\| = \|f^*\|$, respectively.

COROLLARY 7. *If B is a strongly rearrangement invariant Banach space of a.e. φ-continuous functions on $J = (0, 1)$ (where φ is as in Theorem 6) on which there is a strongly rearrangement invariant norm inducing a topology stronger than convergence in measure, then*

$$B \subseteq \Lambda_\varphi(J) \text{ and } \exists A > 0 \text{ such that } \lambda_\varphi(f) \leq A\|f\|, \quad \forall f \in B.$$

PROOF. $\Lambda_J^+ = B \cap \mathfrak{M}^+(J)$ and $\lambda_J(f) = \|f\|$ for $f \in \Lambda_J^+$ provide a pair to which Theorem 6 is applicable. [The completeness property requires the observation that if $f_n \in \Lambda_J^+$, $n = 1, 2, \ldots,$ and $\sum_n \|f_n\| < +\infty$, then $F = \sum_n f_n \in B$ and the summation can also be interpreted pointwise a.e. since norm convergence $\Rightarrow$ convergence in measure.] Thus for each $f \in B$: $f^* \in \Lambda_J^+ \subseteq \Lambda_\varphi^+(J)$ and so $f \in \Lambda_\varphi(J)$. Similarly $\exists A > 0$ for which $\lambda_\varphi(f) = \lambda_\varphi(f^*) \leq A\|f^*\| = A\|f\|$.

Note. With additional continuity assumptions on the T_h this corollary could be established directly by appropriate modifications of the arguments of Stein. (See [**12**].)

REMARK 10. The preceding one dimensional results admit some multidimensional extensions. Denoting the φ of Theorem 6 by φ_1, it is straightforward to verify that Theorem 6 holds in dimension d when $\varphi(x) = \varphi_1(x_1)$ in $J = (0, 1)^d$ and is zero otherwise, since for $f \in \Lambda^+$, the function $f_J^*(x) = (f_J)^*(x_1)$ in J and zero elsewhere is a rearrangement of f_J and thus the one dimensional techniques apply.

Similarly, when φ_1 increases on J and $\varphi(x) = \varphi_1(|x|)$ while f is a positive radially decreasing function on the unit ball S, which vanishes elsewhere, then analogous techniques in polar coordinates for $0 < h = |x| < 1$, yield the lower estimate

$$M_\varphi f(x) \geqslant a \int_0^1 \varphi_1^*(r^{1/d}) f^*(\omega |x|^d r)\, dr$$

so that

$$(M_\varphi f)^*(\xi) \geqslant a \int_0^1 \varphi_1^*(t^{1/d}) f^*(t\xi)\, dt \qquad (0 < \xi < \omega)$$

for an appropriate dimensional constant a, where ω is the measure of the unit ball. It follows that when φ_1 increases so slowly that $\varphi_1^*(t^{1/d}) \geqslant c\varphi^*(t)$, then $\Lambda_\varphi^+(S)$ (respectively, $\Lambda_\varphi(S)$) are again optimal in the sense of Theorem 6 (respectively, Corollary 7), and the maximal inequality of Theorem 1 is optimal in the sense that the order of the right side is correct. In particular, this is the case when $\varphi_1 = 1$ in (0, 1) which corresponds to the multidimensional Hardy-Littlewood maximal function; however our constant, a, is smaller than that obtained by Walker [**15**].

Efforts to date to extend these results to more rapidly increasing φ_1 have not been successful and it may be that in such cases there is a smaller maximal integral than $\int_0^\infty \varphi^*(t) f^*(t\xi)\, dt$.

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